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Notice in example 4 the sequence is non-linear (the difference between terms are different). That is because it was a **Quadratic n^{th} term**, the n was squared.

Now we will look at finding the n^{th} term. There are two methods.

Generating a Sequence from the n^{th} Term

Example 4: The n^{th} term for a sequence is $2n^2 - 3$.

What are the first 5 terms of the sequence?

Position	1	2	3	4	5
Workings	$2\times(1)^2-3$	$2\times(2)^2-3$	$2\times(3)^2-3$	$2\times(4)^2-3$	$2\times(5)^2-3$
Sequence	-1	5	15	29	47

Apply BIDMAS. As the square is only attached to the n, square the n first then multiply by 2!

The sequence is -1, 5, 15, 29, 47...

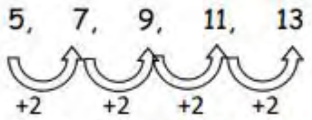
Finding the n^{th} term (linear sequence)

Example 5a: Find the n^{th} term of this sequence.

5, 7, 9, 11, 13

Option 1:

Note the difference between each term



This number goes in front of the n $2n$

Subtract your number from the first sequence number 5 - 2 = 3
This is the second part of your nth term

The n^{th} term is $2n + 3$

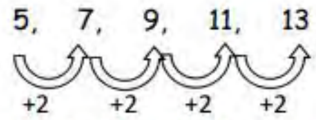
Finding the n^{th} term (linear sequence)

Example 5b: Find the n^{th} term of this sequence.

5, 7, 9, 11, 13

Option 2:

Note the difference between each term



This number goes in front of the n $2n$

Substitute in 1, work out how to get from your number to the first term in the sequence. 2 x 1 = 2 We need 5 so we add 3

The n^{th} term is 2n + 3

Mathematics

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Unit 41

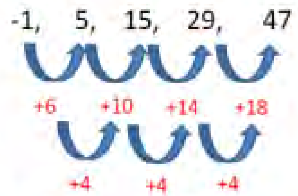
In example 5, the sequence had a common difference.

If the difference between terms changes then it has a quadratic n^{th} term, like example 6.

When you have found the nth term, substitute in some values to check.

Finding the n^{th} term (non-linear/quadratic sequence)

Example 6: Find the nth term of this sequence. **-1, 5, 15, 29, 47**



If you notice the term difference is not constant find the second difference.

Because a second difference is needed, we call the sequence quadratic, and we know the equation will have n^2 in it.

Half the second difference. 4 ÷ 2 = 2 The sequence begins with $2n^2$

As before, subtract the number in front of n from the first term of the sequence (or use the other method). -1 - 2 = -3

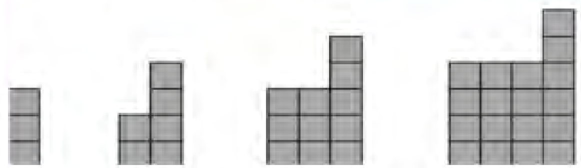
The n^{th} term is $2n^2 - 3$

You may also be asked to find a specific number, for example what is the 50th term in the sequence $4n - 6$.

Substitute in $n = 50$ $4 \times 50 - 6 = 194$

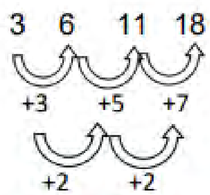
You may also need to apply this knowledge to diagrams, as shown in example 7.

Example 7: How many squares would be in the next diagram?



Convert your diagrams into a numerical sequence
3, 6, 11, 18

Find the n^{th} term



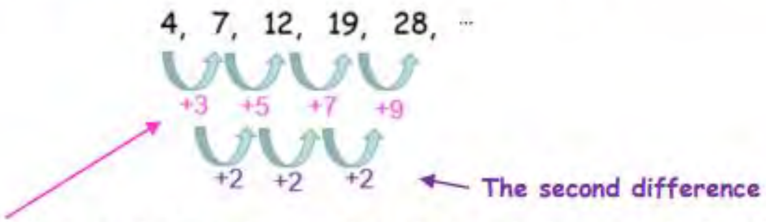
n^{th} term is $n^2 + 2$

Find how many squares are in the 5th diagram

$5^2 + 2 = 27$ squares

Of course, you may have got the nth term by recognising the sequence of square numbers with 2 added squares on top!

Example 8: Find the nth term of this sequence. **4, 7, 12, 19, 28, …**



The sequence does not go up or down in nice equal steps, the steps get bigger each time. Therefore, we find the second difference.

If a second difference is needed , this tells us that the nth term is quadratic (there is an n^2 in it).

Half the second difference. This becomes the coefficient of n^2 (number in front of the n^2).

Half of 2 is 1, but there is no need to write the 1 in front. n^2

Substitute $n = 1, n = 2, n = 3, n = 4, \ldots$ into the equation so far:

$1^2 = 1, \quad 2^2 = 4, \quad 3^2 = 9, \quad 4^2 = 16, \quad 5^2 = 25$

This gives us the sequence: 1, 4, 9, 16, 25, …

Look at what you need to do to this sequence to make the sequence in the question.

1, 4, 9, 16, 25,…

+3 +3 +3 +3 +3

4, 7, 12, 19, 28,…

So, the n^{th} term is: $n^2 + 3$

Mathematics

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Unit 53

Curved Algebraic Graphs

Drawing Curves From their Equations

The equation of a curve is a way of expressing the relationship between the x-coordinates and the y-coordinates that lie on that curve.

Example: $y = x^2 + 3x - 9$

This says that the relationship between all the x-coordinates and all the y-coordinates is: "take the x-coordinate, square it, add on three lots of the x-coordinate, subtract 9, and you get the y-coordinate".

So, if a pair of coordinates such as (2, 1) has this relationship then it lies on the curve. If it does not, such as (5, 4), then it does not lie on the curve.

What you end up with is a curve that goes through all the co-ordinates which share that relationship

Method:

Step 1: If you are not given values of x to use then choose sensible values of x, ones that are small enough to fit on the paper, and easy enough to work out.

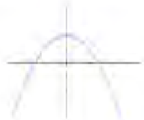
Step 2: Substitute these into the equation to get your y values.

Step 3: Plot the points and join them up with a smooth curve (your pencil should not leave the paper, drawing one continuous curved line)

Check: If the equation has a positive x^2 term, then the graph would have a ∪ shape. If the equation has a negative x^2 term, then the graph would have an ∩ shape.

Eg: $2x^2 + 4x - 3$
Positive x^2 term,
so ∪ shape

Eg: $-x^2 + 5x$
Negative x^2 term,
so ∩ shape



Note: Be careful when substituting negative numbers.

Note: Pick $x = 0$ as one of your points, as it often makes it easier to work out the corresponding y value.

Substituting Numbers without a Calculator

If you are asked to draw a curve on a non-calculator paper then remember:

1. What order you must do operations – remember BIDMAS/BODMAS
2. The rules of negative numbers

Example: Substitute $x = -2$ into $y = x^2 - 4x + 2$

Replace the x terms with -2: $y = (-2)^2 - 4 \times -2 + 2$

Remembering BIDMAS/BODMAS do the squared term first: $y = 4 - 4 \times -2 + 2$

Then the multiplication: $y = 4 - -8 + 2$

The two minus signs together make a plus: $y = 4 + 8 + 2$

Work out the final answer: $y = 14$

So, the point you need to plot has the co-ordinates **(-2, 14)**

Substituting Numbers with a Calculator

Remember:

1. Put your negative numbers in brackets
2. Always do each calculation twice to make sure you did not press a wrong button.

Mathematics

Higher

Unit 53

Example 1:

$y = 2x^2 - 5x$

x	-2	-1	0	1	2	3	4
y		7	0	-3	-2		12

a) Complete the table above.

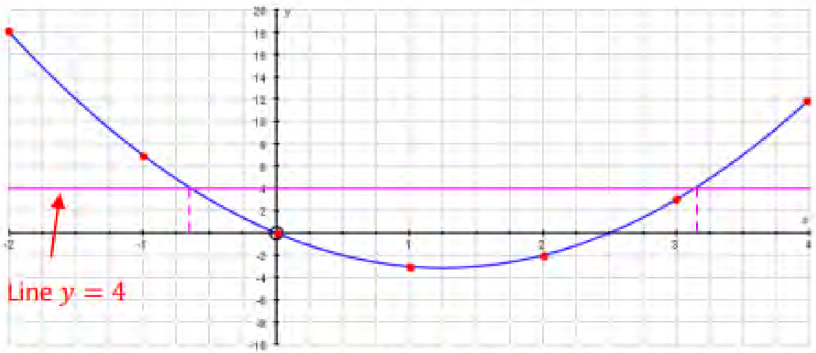
When $x = -2$, $y = 2 \times (-2)^2 - 5 \times (-2) = 18$

When $x = 3$, $y = 2 \times 3^2 - 5 \times 3 = 3$

b) Draw the graph of $y = 2x^2 - 5x$

$y = 2x^2 - 5x$

x	-2	-1	0	1	2	3	4
y	18	7	0	-3	-2	3	12



c) Draw the line $y = 4$ on the graph. Write down the values of x where the line $y = 4$ cuts the curve $y = x^2 - 3x - 4$.

(Where the line crosses the curve, read the corresponding x values)

Values of x are -0.7 and 3.2

Example 2:

$y = x^2 - 3x - 4$

x	-2	-1	0	1	2	3	4	5
y	6	0	-4	-6	-6	-4	0	6

a) Complete the table above. The top line of the table represents the values for x. These need to be substituted into the equation to work out y.

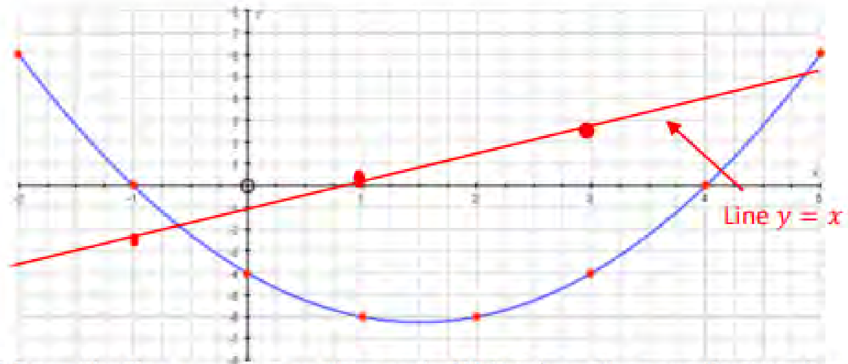
When $x = 0$, $y = 0^2 - 3 \times 0 - 4 = -4$

When $x = 5$, $y = 5^2 - 3 \times 5 - 4 = 6$

b) Draw the graph of $x^2 - 3x - 4$.

$y = x^2 - 3x - 4$

x	-2	-1	0	1	2	3	4	5
y	6	0	-4	-6	-6	-4	0	6



b) Draw the line $y = x - 2$ on the graph. Write down the coordinates where the line $y = x - 2$ cuts the curve $y = x^2 - 3x - 4$.

Substitute values of x into $y = x - 2$ in order to plot points and draw the line. Use a minimum of 3 values. E.g. when x = -2, x = 1 and x = 3. These should make a straight line.

For x = -2, y = -2 -2 = -4 Plot (-2, -4) For x = 1, y = 1 -2 = -1 Plot (1, -1)

For x = 3, y = 3 -2 = 1 Plot (3, 1)

Coordinates where the line crosses the curve are approximately (-0.5, -2) and (4.8, 6).

Mathematics

Higher

Unit 53

Finding the Gradient of Curved Graphs

To find the gradient of the line, we first draw a tangent at the point given.

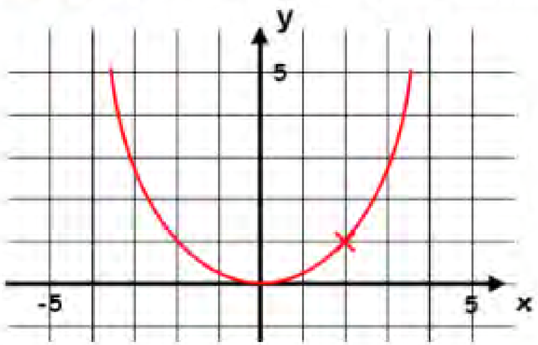
Then, use the formula:

$gradient = \frac{change\ in\ y}{change\ in\ x}$ or $gradient = \frac{y2-y1}{x2-x1}$

Where y_2 and y_1 are the coordinates on the y-axis.

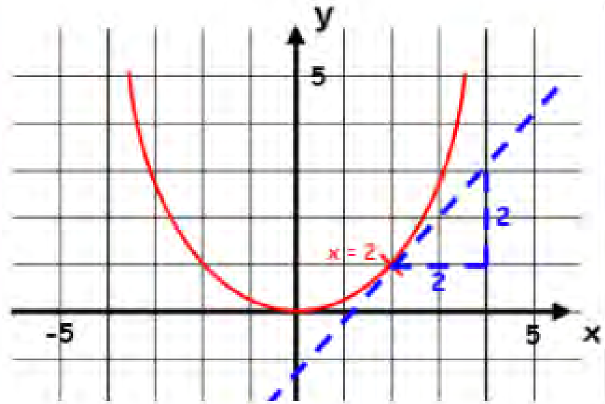
Where x_2 and x_1 are the coordinates on the x-axis.

Example: Find the gradient of the curve at the point $x = 2$.



Draw a straight line that just touches the curve where x = 2

- This line is known as a tangent to the curve
- You can calculate the gradient of it like on a straight line graph
- The value will be an estimate of the gradient of the curve at the given point



$$Gradient = \frac{change\ in\ y}{change\ in\ x}\ or\ \frac{y_2 - y_1}{x_2 - x_1}$$

$$Gradient = \frac{2}{2}$$

$$Gradient = 1$$

In this case, the gradient is positive as the tangent is going UP

Mathematics

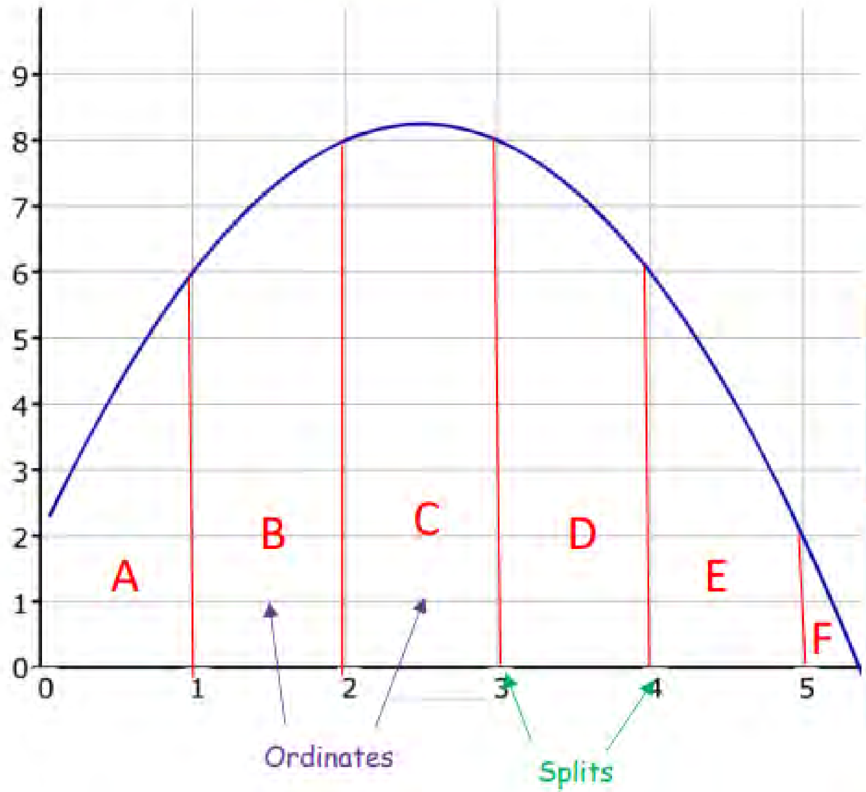
Higher

Unit 53

Finding the Area of Curved Graphs – The Trapezium Rule

We use the Trapezium Rule to find the area under graphs. For a curved graph, this will be an estimate of the area as the lines are not straight. It is called the trapezium rule as we split the curve up into trapeziums (and sometimes some triangles).

Example: Use the trapezium rule to find the area under the curve with **6 ordinates (areas)** or with **5 splits (lines between each area)**. The splits should each have an **equal width**.



Step 1: Split the area up into equal sections (you'll be told how many). Label these A, B, C, etc.

Step 2: Find the area of each section. Remember the formulae you need for area:

Area of a triangle = $\frac{1}{2}$ base x height

Area of trapezium = $\frac{1}{2}(a + b)h$ (where a and b are parallel and h is the perpendicular height)

Step 3: Add the areas together to find the total area.

$Area\ A = \frac{1}{2} \times (a+b) \times h$ $= \frac{1}{2} \times (2+6) \times 1$ $= 4$	$Area\ B = \frac{1}{2} \times (a+b) \times h$ $= \frac{1}{2} \times (6+8) \times 1$ $= 7$	$Area\ C = \frac{1}{2} \times (a+b) \times h$ $= \frac{1}{2} \times (8+8) \times 1$ $= 8$
$Area\ D = \frac{1}{2} \times (a+b) \times h$ $= \frac{1}{2} \times (8+6) \times 1$ $= 7$	$Area\ E = \frac{1}{2} \times (a+b) \times h$ $= \frac{1}{2} \times (6+2) \times 1$ $= 4$	$Area\ F = \frac{1}{2} \times b \times h$ $= \frac{1}{2} \times 2 \times 0.5$ $= 0.5$

Total area = 4 + 7 + 8 + 7 + 4 + 0.5 = 26.5

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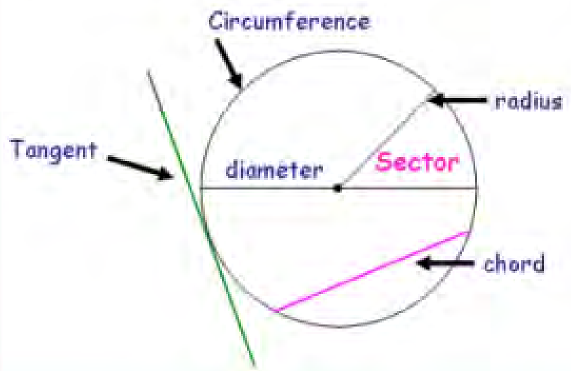
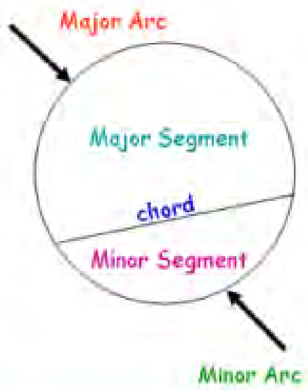
Unit 51

Circle Theorems

Parts of a Circle

It is important we know the names for each part of the circle before we start looking at circle theorems.





Three things you should Learn about Circle Theorems:

1) What each of the theorems say
2) How to spot them
3) How to show you are using circle theorems in your answers

Tips for Answering Circle Theorem Questions

1. Always write down the name of each of the Circle Theorems you have used to get your answer (even if there are more than one)

2. An angle is **not a right-angle** just because it looks like one. You must be able to prove it using a circle theorem or be told it in the question.

3. You will also need to use other angle facts to be able to answer circle theorem questions (See Unit 03 for a recap).

4. Often there are lots of different ways of working out the answer

Mathematics

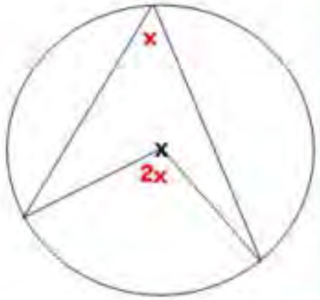
Higher

Unit 51

Theorem 1: Angle at the Centre

Fact: The angle at the centre is twice as big as the angle at the circumference made by the same arc or chord

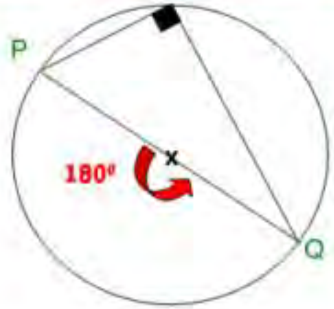
How to spot it: Start with two points (could be the ends of a chord). If you go point-centre-point, the angle you make will be twice as big as if you go point-circumference-point



Theorem 2: Angles in a Semi-Circle

Fact: The angle made at the circumference in a semi circle is a right angle (90°)

How to spot it: Look for a triangle whose base is the diameter of the circle (a line going through the centre). The angle at the circumference in this triangle will always be a right angle

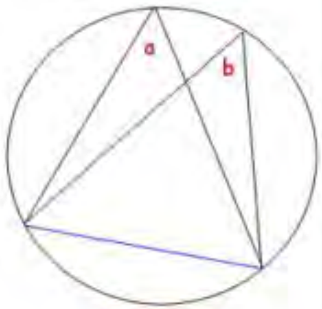


Theorem 3: Angles in the Same Segment

(Or Angles subtended on the same arc)

Fact: Angles in the same segment of a circle are equal to each other

How to spot it: Start with two points (could be the ends of a chord). If you go point-circumference-point, the angle you make will be exactly the same as if you go point-circumference-point, so long as you stay in the same segment of the circle.



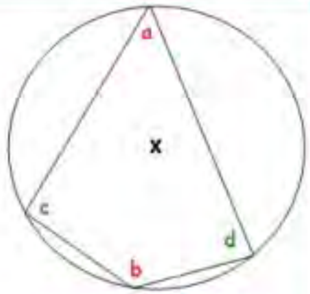
$a = b$

Theorem 4: Cyclic Quadrilateral

Fact: The opposite angles in a cyclic quadrilateral add up to 180°

How to spot it: Look for a four-sided shape with each of the corners on the circumference. The opposite angles in this shape will always add up to 180°

Note: Just like any other quadrilateral, the sum of all the interior angles is still 360°



$a + b = 180°$

$c + d = 180°$

Mathematics

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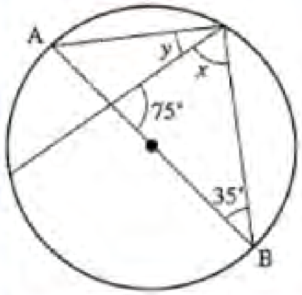
Unit 51

Example 1:



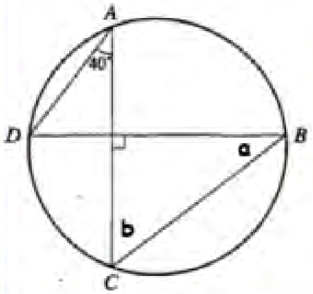
x = 180 - 75 - 35 = 70°

(angles in a triangle)

y = 90 - 70 = 20°

(Theorem 2 - angles in a semi-circle)

Example 2:



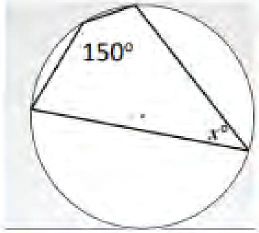
a = 40°

(Theorem 3 - angles in the same segment)

b = 180 - 90 - 40 = 50°

(angles in a triangle)

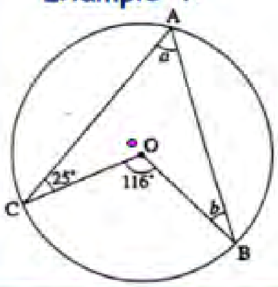
Example 3:



x = 180 - 150 = 30°

(Theorem 4 - angles in a cyclic quadrilateral)

Example 4:



a = 88°

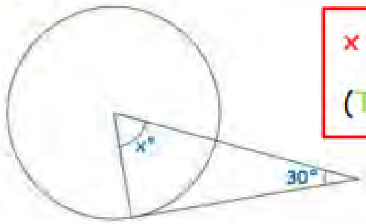
(Theorem 1 – angle at the centre)

To work out b:

● = 360 - 116 = 244° (angles around a point)

b = 360 - 244 - 25 - 88 = 3° (angles in a quadrilateral)

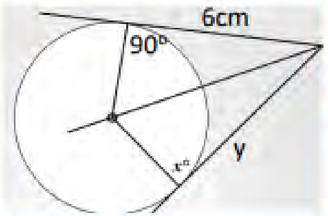
Example 5:



x = 180 - 30 - 90 = 40°

(Theorem 5 – angles on a tangent)

Example 6:



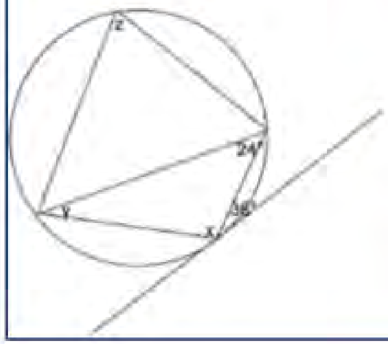
x = 90°

(Theorem 5 – angles on a tangent)

y = 6cm

(Theorem 6 – Two tangents)

Example 7:



y = 36°

(Theorem 6 – alternate segment)

x = 180 – 36 – 24 = 120°

(angles in a triangle)

z = 180 – 120 = 60°

(Theorem 4 – cyclic quadrilateral)

Mathematics

Higher

Unit 2

A Power of a Power

Using index notation: $(a^m)^n = a^{m \times n}$

What it means: Whenever you have a base and it's power raised to another power, you simply multiply the powers together but keep the base the same!

Numbers: If there is a number IN FRONT of the base, then you must raise that number to the power

Examples

$(x^5)^3 = x^{15}$ ✓ — Classic wrong answer: x^8 ✗

$(2^3)^2 = 2^6$ ✓ — Classic wrong answer: 4^6 ✗

$(3a^4)^3 = 27a^{12}$ ✓ — Classic wrong answer: $9a^{12}$ ✗

$(2a^3b^2c)^5 = 32a^{15}b^{10}c^5$ ✓

Zero Index

Using index notation: $a^0 = 1$

What it means: Anything to the power of zero is 1!

Examples $x^0 = 1$ $17^0 = 1$ $5x^0 = 5 \times 1 = 5$

Fractional Indices

Using index notation: $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$

What it means: When a power is a fraction it means you take the root of the base and which root you take depends on the number on the denominator of the fraction, the numerator of the fraction is the power you raise the answer to.

Examples

$64^{\frac{1}{2}} = \sqrt{64} = 8$ The power of a half means the square root

$27^{\frac{1}{3}} = \sqrt[3]{27} = 3$ The power of a third means the cube root

$32^{\frac{1}{5}} = \sqrt[5]{32} = 2$ Because $2^5 = 32$

When the numerator is 1 you do not need to raise the answer to a power ($\sqrt{64}$ is the same as $(\sqrt{64})^1$).

$8^{\frac{2}{3}} = (\sqrt[3]{8})^2 = 2^2 = 4$

Negative Indices

Using index notation: $a^{-m} = \frac{1}{a^m}$

What it means: A negative sign in front of a power is the same as writing "one divided by the base and power". This is called the RECIPROCAL

Note: Only the power and base are flipped over, nothing else!

Examples $x^{-2} = \frac{1}{x^2}$ $5^{-4} = \frac{1}{5^4}$ $5a^{-3} = \frac{5}{a^3}$

$(\frac{1}{3})^{-1} = (\frac{3}{1})^1 = 3$ $(\frac{1}{4})^{-2} = (\frac{4}{1})^2 = 16$ $(\frac{2}{3})^{-3} = (\frac{3}{2})^3 = \frac{27}{8}$

Negative Fractional Indices

Using index notation: $a^{-\frac{m}{n}} = (\sqrt[n]{a})^{-m} = \frac{1}{(\sqrt[n]{a})^m}$

What it means: This is just a mix of the negative indices rule and the fractional indices rule

Examples

$16^{-\frac{1}{2}} = (\sqrt{16})^{-1} = \frac{1}{\sqrt{16}} = \frac{1}{4}$

$8^{-\frac{2}{3}} = (\sqrt[3]{8})^{-2} = \frac{1}{(\sqrt[3]{8})^2} = \frac{1}{2^2} = \frac{1}{4}$

Mathematics

Higher

Unit 9

Ordering Fractions, Decimals and Percentages

To order a mix of fractions, decimals, and percentages you need to first convert all the numbers to the same form, either fractions, decimals, or percentages.

Note: Ascending Order means smallest to largest.

Descending Order means largest to smallest.

Here are some equivalent fractions, decimals, and percentages you should know.

F	D	P
$\frac{1}{100}$	0.01	1%
$\frac{1}{10}$	0.1	10%
$\frac{1}{5}$	0.2	20%
$\frac{1}{4}$	0.25	25%
$\frac{1}{2}$	0.5	50%
$\frac{3}{4}$	0.75	75%
$\frac{1}{3}$	$0.\dot{3}$	$33.\dot{3}\%$
$\frac{2}{3}$	$0.\dot{6}$	$66.\dot{6}\%$

Example:

Put the following in ascending order

56% $\frac{3}{4}$ 0.871   23% $\frac{6}{7}$

To order these, convert them all to decimals.

56% $\frac{3}{4}$ 0.871   23% $\frac{6}{7}$

0.56 (2)   0.75 (3)   0.871 (5)   0.23 (1)   0.857… (4)

Then write the correct order but as they were in the original question.

23%   56% $\frac{3}{4}$ $\frac{6}{7}$ 0.871

Recurring Decimals

Some decimals **terminate**, which means the decimals do not recur, they just stop. For example, 0.75.

A **recurring decimal** exists when decimal numbers repeat forever.

Convert $\frac{8}{11}$ into a decimal using your calculator. A calculator displays this as $0.\dot{7}\dot{2}$ or 0.7272727272…..

The digits 2 and 7 repeat infinitely. This is an example of a **recurring decimal**.

We can show this by writing dots above the 7 and the 2 (the numbers that recur).

If you had to convert into a recurring decimal without the calculator, you would need to use the bus shelter method   Write $\frac{5}{6}$ as a decimal $6\overline{)5.0000}$ = 0.8333   So, $\frac{5}{6} = 0.8\dot{3}$

Here are some more examples of recurring decimals:

$\frac{4}{9} = 0.\dot{4}$ This decimal is made up of an infinite number of repeating 4s.

$\frac{5}{6} = 0.8\dot{3}$ This decimal starts with an 8 and is followed by an infinite number of repeating 3s.

$\frac{2}{7} = 0.\dot{2}8571\dot{4}$ In this decimal, the six digits 285714 repeat an infinite number of times in the same order.

$\frac{9}{22} = 0.4\dot{0}\dot{9}$ This decimal starts with a 4. The two digits 09 then repeat an infinite number of times.

Mathematics

Higher

Unit 55

Simplifying Single Surds

You need to make the number under the square root sign as small as possible

Method

1. Split up the number being square-rooted into a product of at least one square number
2. Use Rule 1 to simplify your answer

Remember: Square Numbers: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100...

Example 1: Simplify $\sqrt{50}$

Split up 50. We ask ourselves: "which square number is a factor of 50?" = 25

$$50 = 25 \times 2$$

HINT: Put the square number first.

So, using Rule 1:

$$\sqrt{50} = \sqrt{25} \times \sqrt{2}$$

We know that $\sqrt{25} = 5$

$$= 5\sqrt{2}$$

Invisible x sign. No need to write it in.

Example 2: Simplify $\sqrt{45}$

Split up 45. We ask ourselves: "which square number is a factor of 45?" = 9

$$45 = 9 \times 5$$

HINT: Put the square number first.

So, using Rule 1:

$$\sqrt{45} = \sqrt{9} \times \sqrt{5}$$

We know that $\sqrt{9} = 3$

$$= 3\sqrt{5}$$

Mathematics

Higher

Unit 55

Simplifying More Than One Surd (Multiplying)

Example: Simplify $\sqrt{90} \times \sqrt{20}$

Let's deal with each surd individually and split them up exactly like we did in the previous section:

$$90 = 9 \times 10 \qquad 20 = 4 \times 5$$

$$\sqrt{90} = \sqrt{9} \times \sqrt{10} \qquad \sqrt{20} = \sqrt{4} \times \sqrt{5}$$

$$\sqrt{90} = 3 \times \sqrt{10} = 3\sqrt{10} \qquad \sqrt{20} = 2 \times \sqrt{5} = 2\sqrt{5}$$

So: $\sqrt{90} \times \sqrt{20} = 3\sqrt{10} \times 2\sqrt{5}$

To simplify further we multiply our whole numbers and our surds separately

$$3 \times 2 = 6 \quad \text{and} \quad \sqrt{10} \times \sqrt{5} = \sqrt{50}$$

So: $3\sqrt{10} \times 2\sqrt{5} = 6\sqrt{50}$

And if you wanted to be really clever, we can simplify even further

$$\sqrt{50} = \sqrt{25} \times \sqrt{2} = 5\sqrt{2}$$

So: $6\sqrt{50} = 6 \times 5\sqrt{2} = 30\sqrt{2}$

Mathematics

Higher

Unit 55

There is more than one way of simplifying surds when **multiplying**.

Other methods:

Example 1: Simplify $\sqrt{90} \times \sqrt{20}$

$= \sqrt{90 \times 20}$ (Use rule 1)

$= \sqrt{1800}$

$= \sqrt{100 \times 18}$

$= \sqrt{100 \times 9 \times 2}$ (Look for square numbers which go into 1800.)

$= \sqrt{100} \times \sqrt{9} \times \sqrt{2}$

$= 10 \times 3 \times \sqrt{2}$

$= 30\sqrt{2}$

Example 2: Simplify $\sqrt{90} \times \sqrt{20}$

$= \sqrt{9 \times 10} \times \sqrt{2 \times 10}$

$= \sqrt{9} \times \sqrt{10} \times \sqrt{2} \times \sqrt{10}$ (I noticed that 10 goes into both 90 and 20.)

$= \sqrt{9} \times \sqrt{10} \times \sqrt{10} \times \sqrt{2}$

$= 3 \times 10 \times \sqrt{2}$ (Changed the order to get the $\sqrt{10}$'s together.)

$= 30\sqrt{2}$ (Rule 2 says that $\sqrt{10} \times \sqrt{10} = 10$)

Simplifying More Than One Surd (Dividing)

Good News: Do these in exactly the same way

Example: Simplify $\frac{\sqrt{60} \times \sqrt{20}}{\sqrt{12}}$

$= \frac{\sqrt{60 \times 20}}{\sqrt{12}}$ (Use rule 1)

$= \frac{\sqrt{120}}{\sqrt{12}}$

$= \sqrt{\frac{120}{12}}$ (Use rule 3)

$= \sqrt{10}$

This is the final answer. Even though you have probably noticed that 5 and 2 go into 10, neither 5 nor 2 are square numbers so we wouldn't be able to simplify it any more.

Mathematics

Higher

Unit 55

Simplifying More Than One Surd (Adding and Subtracting)

We can only add and subtract surds of the same type

So, we must use our simplifying skills to change them into the same type.

$4\sqrt{3} + 5\sqrt{3} = 9\sqrt{3}$ (in the same way we would do 4a + 5a = 9a)

$10\sqrt{5} - 3\sqrt{5} = 7\sqrt{5}$

$2\sqrt{7} + 8\sqrt{6}$ We can't simplify this because the numbers under the roots are different (in the same way we can't simplify 2a + 8b).

Example 1: Simplify $\sqrt{12} + \sqrt{27}$

The answer is definitely NOT: $\sqrt{39}$

We need to simplify the surds first:

$12 = 4 \times 3$

$\sqrt{12} = \sqrt{4} \times \sqrt{3}$

$\sqrt{12} = 2 \times \sqrt{3} = 2\sqrt{3}$

$27 = 9 \times 3$

$\sqrt{27} = \sqrt{9} \times \sqrt{3}$

$\sqrt{27} = 3 \times \sqrt{3} = 3\sqrt{3}$

So: $\sqrt{12} + \sqrt{27} = 2\sqrt{3} + 3\sqrt{3}$

Our surds are now of the same type. Each term has $\sqrt{3}$ in it.

We can now just add our whole numbers. $2\sqrt{3} + 3\sqrt{3} = 5\sqrt{3}$

Example 2: Simplify $\sqrt{63} - \sqrt{28}$

Simplify the surds:

$63 = 9 \times 7$

$\sqrt{63} = \sqrt{9} \times \sqrt{7}$

$\sqrt{63} = 3 \times \sqrt{7} = 3\sqrt{7}$

$28 = 4 \times 7$

$\sqrt{28} = \sqrt{4} \times \sqrt{7}$

$\sqrt{28} = 2 \times \sqrt{7} = 2\sqrt{7}$

So: $\sqrt{63} - \sqrt{28} = 3\sqrt{7} - 2\sqrt{7}$

Our surds are now of the same type. Each term has $\sqrt{7}$ in it.

We can now just subtract our whole numbers. $3\sqrt{7} - 2\sqrt{7} = \sqrt{7}$

Mathematics

Higher

Unit 55

Expanding Brackets with Surds

Rule: Use FOIL to multiply out the brackets as you would in algebra (multiply every term in the first bracket by every term in the second bracket).

Example 1: $(3+\sqrt{5})(6+\sqrt{5})$

First: $3 \times 6 = 18$

Outside: $3 \times \sqrt{5} = 3\sqrt{5}$

Inside: $\sqrt{5} \times 6 = 6\sqrt{5}$

Last: $\sqrt{5} \times \sqrt{5} = 5$

So: $(3+\sqrt{5})(6+\sqrt{5}) = 18 + 3\sqrt{5} + 6\sqrt{5} + 5$

$= 23 + 9\sqrt{5}$

Simplifying the whole numbers gives

$18 + 5 = 23$

Simplifying the surds gives

$3\sqrt{5} + 6\sqrt{5} = 9\sqrt{5}$

(We can add here as the surds are the same)

Example 2: $(\sqrt{2}+7)(4-\sqrt{8})$

First: $\sqrt{2} \times 4 = 4\sqrt{2}$

Outside: $\sqrt{2} \times -\sqrt{8} = -\sqrt{16} = -4$

Inside: $7 \times 4 = 28$

Last: $7 \times -\sqrt{8} \quad = -7\sqrt{8} \quad = -7\sqrt{4} \times \sqrt{2}$

$= -7 \times 2 \times \sqrt{2} \quad = -14\sqrt{2}$

So: $(\sqrt{2}+7)(4-\sqrt{8}) = 4\sqrt{2} - 4 + 28 - 14\sqrt{2}$

$= 24 - 10\sqrt{2}$

Simplifying the whole numbers gives

$-4 + 28 = 24$

Simplifying the surds gives

$4\sqrt{2} - 14\sqrt{2} = -10\sqrt{2}$

(We can subtract here as the surds are the same)

Mathematics

Higher

Unit 43

Probability

Probability is the likelihood that an event will occur.

Probabilities are always written as fractions, decimals, or percentages.

Probabilities have values between 0 and 1.

Probability scale

Probabilities can be described using words

Impossible — Unlikely — Even Chance — Likely — Certain

What is the probability that if you toss a coin it will land on heads?
Even Chance (the coin is equally likely to land on heads or tails)

What is the probability that Winter follows Summer?
Impossible

What is the probability that Christmas will be on the 25th of December?
Certain

Probabilities can also be described using numbers

Impossible	Unlikely	Even	Likely	Certain
0		1/2		1
0		0.5		1.0
0%		50%		100%

The probability of an event happening can be found using:

$$\text{P(event happening)} = \frac{\text{number of ways the event could happen}}{\text{the total number of outcomes}}$$

Example 1: Find the probability of throwing an even number on a dice.

$$\text{P(even number)} = \frac{3}{6}$$

3 ← Number of even numbers on a dice (2, 4, 6)

6 ← Total amount of numbers on a dice

Example 2: What is the probability of picking a diamond from a full deck of cards?

$$\text{P(diamond)} = \frac{13}{52}$$

13 ← Number of diamonds in a pack of cards

52 ← Total number of cards in a pack of cards

The probability of an event **not** happening can be found using:

P(event not happening) = 1 – P(event happening)

Example: What is the probability of **not** picking a diamond from a full deck of cards?

P(not diamond) = 1 – P(diamond)

$$= 1 - \frac{13}{52} \quad = \frac{39}{52}$$

Mathematics

Higher

Unit 43

Sample Space Diagram

A sample space diagram is a way of showing multiple outcomes in one diagram.

Example: Two dice are thrown, and the numbers are multiplied together. The table below shows some of the possible outcomes.

First dice x second dice
5 x 6 = 30

Second Dice	First Dice: 1	2	3	4	5	6
6	6	12	18	24	30	36
5	5	10	15	20	25	30
4	4	8	12	16	20	24
3	3	6	9	12	15	18
2	2	4	6	8	10	12
1	1	2	3	4	5	6

First dice x second dice
6 x 3 = 18

a) Complete the table to show **all** the possible outcomes.

b) What is the probability of getting an outcome that is an odd number?

P(odd) = $\frac{9}{36} = \frac{1}{4}$

Number of outcomes that are odd numbers

Total number of outcomes

c) If the two dice were thrown a total of 60 times, how many times would you expect to get an outcome greater than 10?

P(6 **and** H) = 60 x $\frac{9}{36}$ = 15 times

Number of times the dice are thrown

Probability of an odd number

Listing Outcomes

You might be asked to list all the possible outcomes for two or more events.

Example: List all the 3-digit numbers that can be made using the digits 3, 6, and 9?

369 396 639 693 936 963

Example: A coin is flipped, and a dice is rolled. List all the possible outcomes.

H 1 H 2 H 3 H 4 H 5 H 6

T 1 T 2 T 3 T 4 T 5 T 6

A head on the coin → H 1; A 1 on the dice → H 1; A tail on the coin → T 6; A 6 on the dice → T 6

Finding Missing Probabilities from a Table

Probabilities add up to 1, to find the missing probabilities add together the probabilities you are given and subtract them from 1.

Example: A biased spinner has 4 colours. The probability of the spinner landing on each colour is given below.

Colour	Red	Blue	Yellow	Green
Number of times	0.1	x	0.4	0.2

a) What is the probability of choosing a blue sweet?

P(Blue) = 0.1 + 0.4 + 0.2 = 0.7 ← Add the probabilities

1 - 0.7 = 0.3 ← Subtract them from 1

b) The spinner is spun 100 times. Calculate an estimate for the number of times the spinner will land on yellow.

P(Yellow) = 100 x 0.4 = 40 times

Number of times the spinner is spun

Probability of a yellow

Mathematics

Higher

Unit 43

The 'AND' / 'OR' rules

Key words

Independent events – one event happening does not change the probability of the other one happening

Mutually exclusive events – events that are not able to happen at the same time as each other

The 'AND' rule

If you want one outcome **and** the other outcome, then you multiply their probabilities

For two independent events A and B

P(A **and** B) = P(A) **x** P(B)

Example:

Find the probability of rolling a 6 on a dice **and** getting a head on the toss of a coin

P(6 **and** H) $= \frac{1}{6} \times \frac{1}{2}$

$= \frac{1}{12}$

The 'OR' rule

If you want one outcome **or** the other outcome, then you add their probabilities

For two mutually exclusive events A and B

P(A **or** B) = P(A) + P(B)

Example:

The table below shows the probability that a spinner lands on a certain colour

Colour	Yellow	White	Blue	Red
Probability	0.2	0.25	0.15	0.4

What is the probability that the spinner lands on yellow or red?

P(Y **or** R) = 0.2 + 0.4

= 0.6

Relative Frequency

Some probabilities can be estimated by doing experiments or trials, this is called relative frequency.

The more trials that are done (100+), the more accurate the estimated probability will be.

$$\text{Relative Frequency} = \frac{\text{number of times the event occurs}}{\text{total number of trials}}$$

Relative frequency from a table

Example:

A spinner is spun 100 times. The colour on the spinner is recorded after each spin. The table below shows the results recorded.

Colour	White	Green	Blue
Frequency	21	52	27

What is the relative frequency of spinning a green?

$\frac{52}{100} = 0.52$

Total number of spins

Mathematics

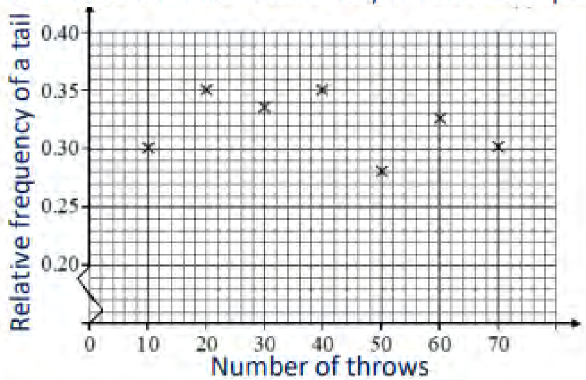
Higher

Unit 43

Relative frequency from a graph

Example:

A coin is thrown 70 times. The relative frequency of the number of tails after every 10 throws is plotted.



a) How many tails were obtained in 50 throws?

0.28 x 50 = 14 tails

Relative frequency of 50 — Number of throws

b) Use the diagram to estimate the probability of getting a tail.

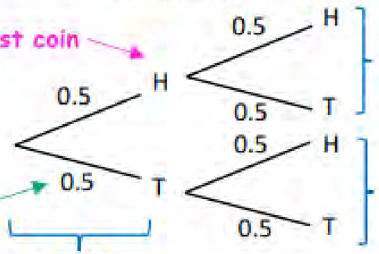
P(T) = 0.3 (the more throws the more accurate the result, 70 throws = 0.3)

Probability Trees

Tree diagrams are a way of showing combinations of two or more events.

Tree diagrams have branches, with each branch adding to 1.

Here is a tree diagram showing the outcomes of two coins being thrown.

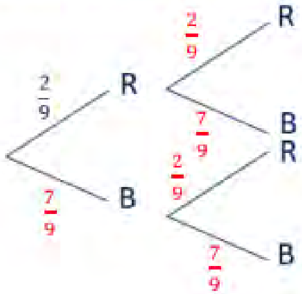


Example:

A bag contains red and blue counters. The probability that a red counter is chosen is $\frac{2}{9}$.
A counter is chosen and replaced; a second counter is chosen.

a) Complete the tree diagram below.

remember each branch adds to 1



b) Calculate the probability that a red counter is chosen followed by a blue counter.

We need to follow the branches, red for the first counter **AND** blue for the second counter.

P(R **and** B) = $\frac{2}{9} \times \frac{7}{9} = \frac{14}{81}$

c) Calculate the probability that two counters of the same colour are chosen.

P(R **and** R) = $\frac{2}{9} \times \frac{2}{9} = \frac{4}{81}$

OR

P(B **and** B) = $\frac{7}{9} \times \frac{7}{9} = \frac{49}{81}$

$\frac{4}{81} + \frac{49}{81} = \frac{53}{81}$

Mathematics

Higher

Unit 43

Conditional Probability

Conditional Probability

Conditional probability is when the probability of an event occurring depends on the outcome of another event.

For example, the probability that I take an umbrella to work could depend on the probability of it raining on a given day.

Example 1: A bag contains 4 red balls and 1 blue ball. A ball is removed from the bag at random and its colour is noted. The ball is NOT replaced in the bag. Another ball is chosen at random.

What is the probability that the second ball selected is red?

This is conditional probability because the chance of getting a red ball the second time depends on which ball was chosen the first time. If a red ball has already been removed there are only 3 red balls left whereas if a blue ball was chosen the first time, then all of the balls left are red. We need to work out both possible scenarios.

P(blue then red) = P(blue first) x P(red second)

$= \frac{1}{5} \times \frac{4}{4}$ ← Once 1 blue has been chosen, there are only 4 balls left in the bag.

$= \frac{1}{5} \times 1$

$= \frac{1}{5}$

P(red then red) = P(red first) x P(red second)

$= \frac{4}{5} \times \frac{3}{4}$ ← Once 1 red has been chosen, there are only 3 red balls left and only balls left in the bag.

$= \frac{12}{20}$

$= \frac{3}{5}$

P(second is red) $= \frac{1}{5} + \frac{3}{5} = \frac{4}{5}$

Example 2: The probability that Tom catches the bus to school is 0.7. If Tom catches the bus to school, the probability that he is is late for school is 0.2. If Tom does not catch the bus, the probability that he's late is 0.4.

What is the probability that Tom is late for school on any given day?

Again, we need to work out both possible scenarios.

P(catches bus and late) = P(catches bus) x P(late)

= 0.7 x 0.2

= 0.14

P(catches bus and late) = P(doesn't catch bus) x P(late)

= 0.3 x 0.4

= 0.12 ← P(doesn't catch bus) = 1 - 0.7

P(late) = 0.14 + 0.12 = 0.26

Mathematics

Higher

Unit 22

Drawing Straight Line Graphs from their Equation

As well as graphs of horizontal and vertical lines, there are also graphs of diagonal lines.

Method for Drawing Straight-Line Graphs

1. If the question does not give you values of x to use, then choose sensible values of x (A good choice of x values are 0, 1 and 2. This will show you the direction of the line. You need **at least** 3 values of x but choosing 4 (values -1, 0, 1 and 2) would make it even better).

2. Carefully substitute each x value it into the equation to get your y values, be careful if substituting negative numbers.

3. Join up the points with a straight line

Note: The points should make a straight line, if one of your points does not lie on the straight line, check your substitution again.

Substituting:

$y = 3x - 1$ → 3 x the x-coordinate then – 1

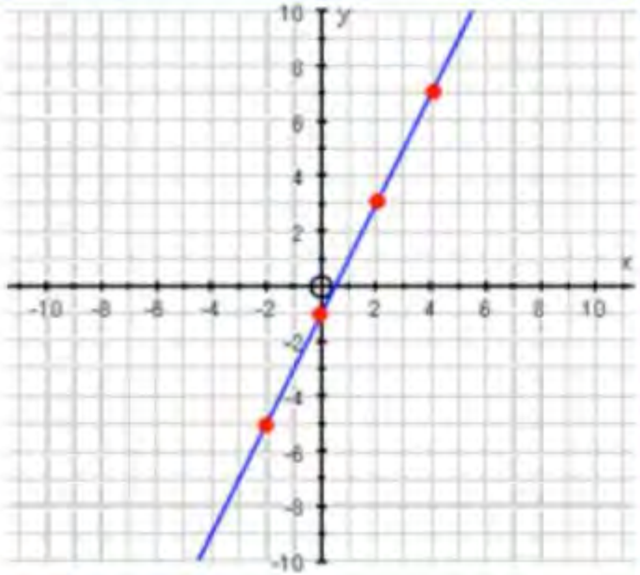
x	-3	0	3
y	-10	-1	8

Draw a table to display this information

This represents a coordinate pair (-3, -10)

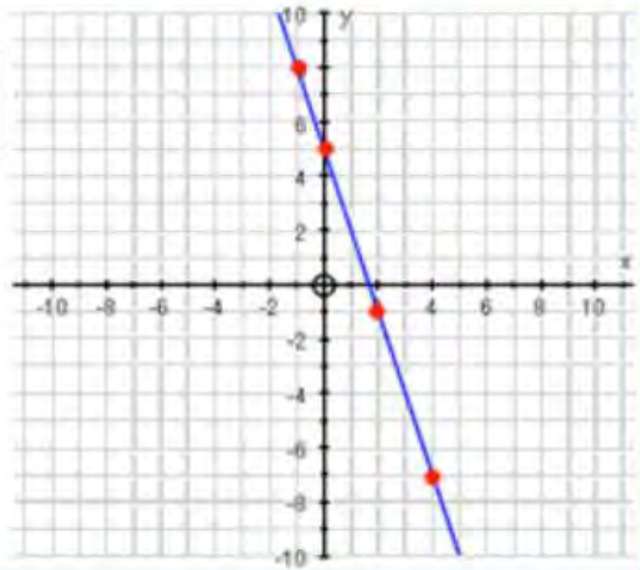
$y = 2x - 1$

x	0	2	4	-2
y	-1	3	7	-5



$y = -3x + 5$

x	0	2	4	-1
y	5	-1	-7	8



Mathematics

Higher

Unit 22

The Equation of a Line in the Form $y = mx + c$

$$y = mx + c$$

The gradient m

- This tells you the gradient/steepness of the line
- The bigger the number, the steeper the line
- If the number is positive, the line slopes upwards
- If it is negative, the line slopes downwards
- Parallel lines have the same gradient

Example: $y = 4x + 7$ has a gradient of 4, $y = -9x + 3$ has a gradient of -9.

The y-intercept c

- This tells you where the line crosses the y axis

Example: $y = 2x + 5$ crosses the y-axis at 5, $y = 2x - 5$ crosses the y-axis at -5.

Note: If the equation is NOT in the form: y = mx + c, you must first re-arrange it.

If you are given points and not a line all you need to do is join the points up to give you a line before you start.

Note: You may be asked to draw a line given an equation which looks like this:

$$9x + 3y = 18.$$

Fist, rearrange the equation into the form $y = mx + c$.

$$9x + 3y = 18$$

$$3y = 18 - 9x$$

$$y = \frac{18}{3} - \frac{9x}{3}$$

$$y = 6 - 3x$$

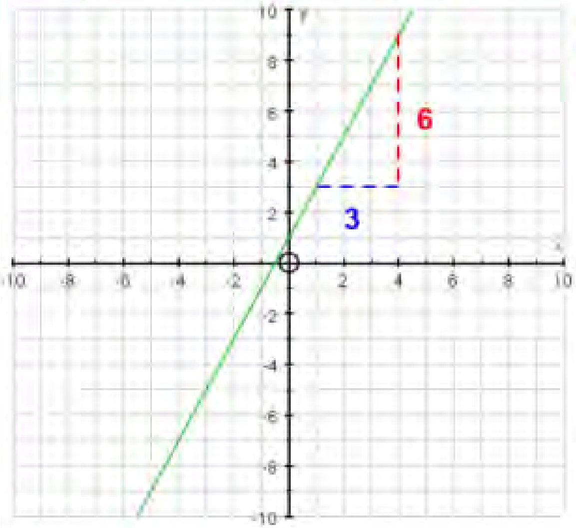
$$y = -3x + 6$$

$$y = mx + c.$$

Then, follow the steps as above.

Working Out the Equation of a Line

Using our knowledge of y = mx + c, we can actually work backwards and figure out the equation of a straight line by its graph.



First, we must work out the gradient of the line by drawing a right-angled triangle anywhere on the line and using this formula:

$$\text{Gradient} = \frac{\text{Change in } y}{\text{Change in } x}$$

$$\text{Gradient} = \frac{6}{3} = 2$$

Then we work out the y-intercept, which is the place the line crosses the y axis, (0, 1)

So, the equation of the line is:

$$y = 2x + 1$$

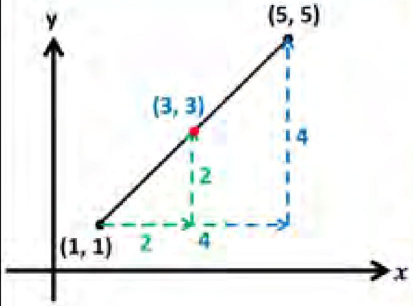
Mathematics

Higher

Unit 22

Midpoint of a Line

The midpoint is exactly half-way between two coordinates



The Midpoint of a Line

$\left(\frac{x_1 + x_2}{2}\right), \left(\frac{y_1 + y_2}{2}\right)$

Add up the x-coordinates and divide by 2	Add up the y-coordinates and divide by 2
$\left(\frac{1+5}{2}\right)$	$\left(\frac{1+5}{2}\right)$
3	3

Equations of Parallel and Perpendicular Lines

Parallel lines *are lines that never meet* - they are always a fixed distance apart. Lines that are parallel will have the same **gradient.**

i.e. The graphs of $y = 2x + 1$ and $y = 2x - 2$ have the same gradient of 2 so they are **parallel**.

Example

Write an equation of a line that is parallel to $y = 7x + 5$

The gradient of the line in the question is **7**. So, any line with a gradient of 7 will be parallel.

Two examples of answers you could give are: $y = 7x - 15$ and $y = 7x + 3.5$

Two lines are ***perpendicular*** *if they meet at a right angle.* Two lines are perpendicular if the product of their gradient is -1.

Example

Find the equation of a straight line that is perpendicular to $y = 3x + 2$

The gradient of $y = 3x + 2$ is 3.

To find the perpendicular gradient you need to find the number which multiplies by 3 to give -1. This is called the negative reciprocal.

The negative reciprocal of 3 is $-\frac{1}{3}$ So, any line with the gradient $-\frac{1}{3}$ will be perpendicular.

i.e. $y = -\frac{1}{3}x + 5$ or $y = -\frac{1}{3}x - 6$

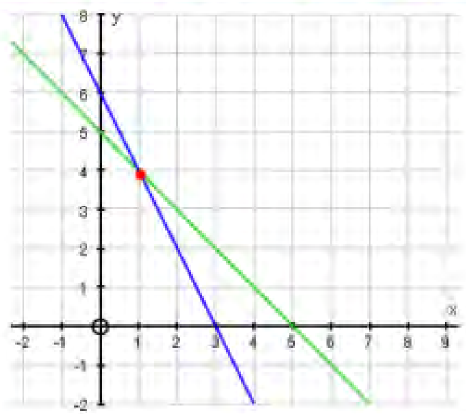
Using Straight Line Graphs to Solve Simultaneous Equations

It is possible to use straight line graphs to solve simultaneous equations

All you need to do it to carefully plot both lines, and the point where they cross is the solution, but remember you want $x =$ and $y =$

Example: Solve the following pair of simultaneous equations graphically:

$x + y = 5$ and $2x + y = 6$



$x + y = 5$

x	0	5
y	5	0

$2x + y = 6$

x	0	3
y	6	0

The solution is $x = 1$ and $y = 4$

Mathematics

Higher

Unit 31

Trigonometry

Just like Pythagoras' Theorem, Trigonometry only works with RIGHT-ANGLED TRIANGLES. However, Trigonometry can be used to find missing sides **or** missing angles.

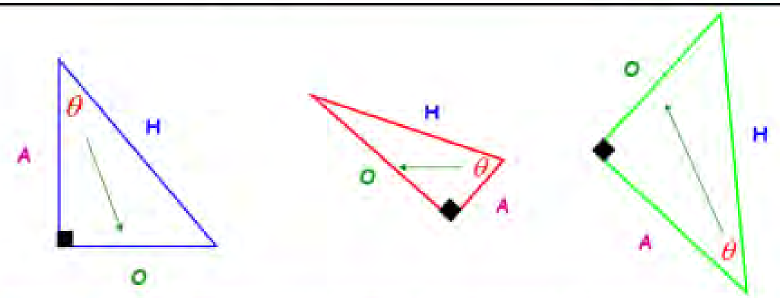
Labelling the Sides of a Right-Angled Triangle

This is the order to do it:

Hypotenuse (H) – the longest side, opposite the right-angle

Opposite (O) – the side directly opposite the angle you have been given / asked to work out

Adjacent (A) – the only side left.



Note: θ is just the Greek letter Theta, and it is used for unknown angles, just like x is often used for unknown lengths.

Sine, Cosine and Tangent (SOH CAH TOA)

Method for finding missing sides or angles:

Step 1: Label your right-angled triangle

Step 2: Tick which information (lengths of sides, sizes of angles) you have been given

Step 3: Tick which information you have been asked to work out

Step 4: Decide whether the question needs **sin**, **cos**, or **tan**

Learn the following formulas:

$$\text{Sine } \theta = \frac{\text{Opposite}}{\text{Hypotenuse}} \qquad \text{Sin } \theta = \frac{O}{H}$$

$$\text{Cosine } \theta = \frac{\text{Adjacent}}{\text{Hypotenuse}} \qquad \text{Cos } \theta = \frac{A}{H}$$

$$\text{Tangent } \theta = \frac{\text{Opposite}}{\text{Adjacent}} \qquad \text{Tan } \theta = \frac{O}{A}$$

Substitute in the two values you do know and re-arrange the equation to find the value you don't know.

A good way to remember the formulae is to use the initials from left to right:

SOH CAH TOA

To decide if you need to use sin, cos, or tan, you could highlight or tick the information you have been given and asked to work out.

Mathematics

Higher

Unit 31

Solving Trigonometry Problems

When finding an angle, remember you need to use one of the inverse operations either sin^{-1}, cos^{-1} or tan^{-1}

You will need to press SHIFT on your calculator first.

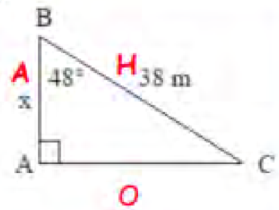
Example 1: Find the length of AB (this can be labelled x)

Step 1: Label the sides O, A and H

Step 2 and 3: Tick or highlight what you have been given and what you have been asked to work out

SOH CAH TOA

Step 4: Decide whether you need sin, cos or tan. Looking above the only one that contains both A and H is **cos**



$$\cos\theta = \frac{A}{H}$$

$$\cos 48 = \frac{x}{38}$$

$$38 \times \cos 48 = x$$

$$x = 25.4m\ (1dp)$$

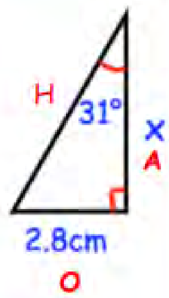
Example 2: Find the length of x

Step 1: Label the sides O, A and H

Step 2 and 3: Tick or highlight what you have been given and what you have been asked to work out

SOH CAH TOA

Step 4: Decide whether you need sin, cos, or tan. Looking above the only that contains both A and O is **tan**



$$\tan\theta = \frac{O}{A}$$

$$\tan 31 = \frac{2.8}{x}$$

Note: Because the unknown x is on the bottom you would need to multiply up by x and divide by $\tan 31$ but it is easier just to swap them.

$$x = \frac{2.8}{tan31}$$

$$x = 4.66cm\ (2dp)$$

(on the calculator you can press 2.8 ÷ tan31 =)

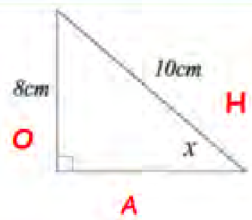
Example 3: Find the size of the angle x

Step 1: Label the sides O, A and H

Step 2: Tick or highlight what you have been given

SOH CAH TOA

Step 4: Decide whether you need sin, cos, or tan. Looking above the only that contains both O and H is **sin**



$$\sin x = \frac{O}{H}$$

$$\sin x = \frac{8}{10}$$

(on the calculator you can press shift sin, 8 ÷ 10 =)

$$x = \sin^{-1}\left(\frac{8}{10}\right)$$

$$x = 53.1°\ (1dp)$$

Do not forget to round your answers to an appropriate degree of accuracy.

Mathematics

Higher

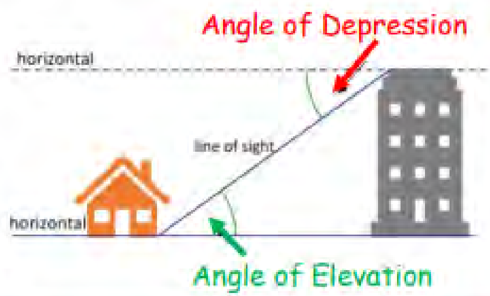
Unit 31

Solving multi-step Trigonometry problems

Some questions involve more than one step, so you may be required to use Trigonometry twice to find a missing angle and/or side. You may even have to use Pythagoras' Theorem in some questions like in example 3.

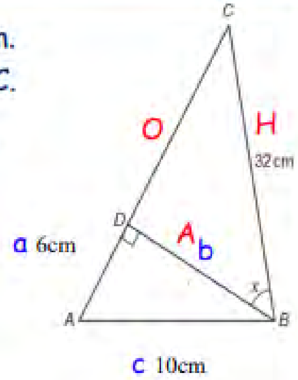
Some questions, like example 4, involve an angle of elevation or an angle of depression.

The Angle of Elevation is formed by looking UP from the horizontal, (stood at the house, looking up at the top of the tower).
The Angle of Depression is formed by looking DOWN from the horizontal, (stood on the top of the tower looking down at the house).



Example 3: AD = 6cm, AB = 10cm and BC = 32cm. D is on the line AC and BD is perpendicular to AC.

Calculate the size of angle x to 1 decimal place.



Looking at the triangle BCD, you will need one more side before you can find angle x. The triangle ABD is right angled so Pythagoras' Theorem can be used to find BD.

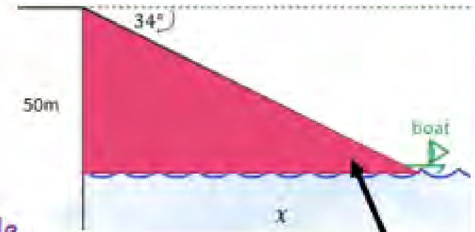
$$b^2 = c^2 - a^2$$
$$= 10^2 - 6^2$$
$$= 100 - 36$$
$$b^2 = 64$$
So, $b = \sqrt{64} = 8$

Therefore BD = 8cm and using Trigonometry angle x can be found.

(on the calculator you can press shift cos, 8 ÷ 32 =)

$$\cos x = \frac{A}{H}$$
$$\cos x = \frac{8}{32}$$
$$x = \cos^{-1}\left(\frac{8}{32}\right)$$
$$x = 75.5^o\ (1dp)$$

Example 4: A man standing on a cliff sees a boat at an angle of depression of 34°. If the cliff is 50m tall, how far from the cliff is the boat?

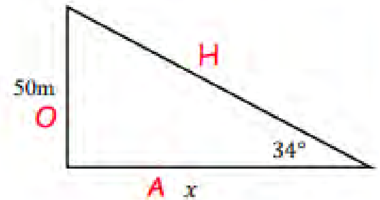


Before we label the sides of the triangle, we need an angle inside the triangle. Using the alternate angles in parallel lines are equal rule, we can determine that this angle is also 34°.

Step 1: Label the sides O, A and H

Step 2 and 3: Tick or highlight what you have been given and what you have been asked to work out

SOH CAH TOA



Step 4: Decide whether you need sin, cos or tan. Looking above the only one that contains both O and A is tan

$$\tan\theta = \frac{O}{A}$$
$$\tan 34 = \frac{50}{x}$$
$$x = \frac{50}{tan34}$$
$$x = 74.13m\ (2dp)$$